

different points of view. As has been said, Stone attributed to the condition an *a priori* character. On the other hand, it appears from a later note * that Schiaparelli started with the necessary differential equations and then sought to give them as it were a physical interpretation. Thus Schiaparelli's proof appears frankly artificial, and confirms a remark, made earlier in the present note, that it may be wiser to regard the hypotheses as defining the properties which the observations must possess rather than as axioms on which a rigorous proof of the principle of the arithmetic mean can be founded.

12. I cannot conclude this brief note without expressing my indebtedness to that section of Emanuel Czuber's "Theorie der Beobachtungsfehler" which treats of the principle of the arithmetic mean, both for references and for a connected account of the main lines of earlier discussions of the subject.

University Observatory, Oxford:
1902 May 28.

Experimental Reduction of Photographs of Eros for the Determination of the Solar Parallax. Second Paper: Combination of Results from Mount Hamilton, Minneapolis and Cambridge.
Arthur R. Hinks, M.A.

1. In a recent paper (*Monthly Notices*, 1901 November, vol. lxii. p. 22) I attempted to develop a somewhat new method of comparing with an ephemeris positions of a planet derived from measures upon photographs. Briefly the method was this: Separate heliocentric ephemerides of the planet and of the Earth are expressed in rectangular coordinates referred to a mean ecliptic and equinox, the unit being naturally the mean distance of the Sun from the Earth. From these two simple ephemerides (which arise in the process of computing the usual apparent place ephemeris) it is comparatively easy to compute the position of the planet upon a "standard" plate of given centre; the only transformation of coordinates which is not a mere addition, viz. the change from axes in and perpendicular to the ecliptic to axes in and perpendicular to the plate, is expressed in terms of direction cosines and performed with considerable ease on an arithmometer. In fact, the method offers the advantage that almost throughout the coordinates are rectangular, the transformations linear, and the numerical operations are consequently such as can be performed much more readily with a calculating machine than with logarithms.

2. For a trial of the method I measured and reduced fifty-two exposures of *Eros* made at Cambridge on 1900 November 9 and

* *Astr. Nachr.* Nr. 2097 (1876).

10. All the details of the reductions to standard, and of the method then used of comparing with the ephemeris computed for me by Professor E. Millosevich, will be found in the first paper. We need mention here only one point. In order satisfactorily to represent the observations it was necessary to correct the ephemeris by terms varying directly with the time. At the same hour angles on two successive nights the differences, computed minus observed position of the planet, were not the same. It seemed that the ephemeris would not represent the motion of the planet for so long a period as twenty-four hours, and in that case it would be clearly impossible to determine the parallax by the diurnal method from the results of any number of single disconnected nights. Parallactic shift and progressive error of the ephemeris are hopelessly interwoven, and nothing can be done unless we have observations at the same hour angles, and therefore with the same parallax factors, on adjacent nights. Then only can the progressive error of the ephemeris be separated from the parallax and empirically eliminated before proceeding to the solution of the equations of condition. It is clear, however, that this procedure breaks down unless the series of observations is fairly continuous—which in the winter of 1900-01 was by no means the case—and that in general a large part of the work of any observatory must be excluded from a parallax discussion of the results of that observatory alone.

The difficulty might be overcome by making a preliminary solution of all the results from all the co-operating observatories, deducing therefrom a series of corrections to the tabular place of the planet, and basing upon these a new ephemeris. This would represent the apparent motion of the planet very accurately. But Sir David Gill remarks, in his discussion of the heliometer observations of *Victoria*, that the resulting ephemeris must be looked upon rather as an interpolation formula than as the equivalent of a genuine orbit that will continue to represent the planet's motion for any length of time; wherefore he took his tabular places from the second ephemeris of *Victoria* and his tabular distances from the first; and there is the further disadvantage that a complete preliminary discussion for tabular errors of the ephemeris would delay for an indefinite time the deduction of any satisfactory value of the parallax; the mass of observations is too large and too heterogeneous to be dealt with in this laborious way.

But we have an obvious alternative. In every case of a pair of nearly simultaneous observations of *Eros* made at stations far apart, the tabular place of *Eros*, be it good or bad, is practically eliminated when it is a question only of finding the parallax, and not of correcting the ephemeris, and it seems that at first it will be better policy to aim at combining directly the observations in groups extending over so short a time that the errors of the ephemeris have no time to develop, rather than to work for a complete general reduction of the whole material at one

sweep. The photographs of *Eros* number so many thousands that we may expect to find a great many cases where the observations may be separated in this way into small groups. The aim of this paper is to apply to two or three such cases the method of reduction which was developed in the previous paper.

3. The publication in the Paris circulars of a detailed statement of what each observatory has achieved makes it very easy to select good examples of pairs of nearly simultaneous observations at stations far apart. I found that at three observatories in the United States—Lick, Minneapolis, and Northfield—they had plates taken very nearly at the same time as some of those of 1900 November 9 and 10 which I was reducing by the diurnal method. A request for early measures of these plates brought from the directors of the three observatories a very kind and ready response. The measurement of the photographs made at Mount Hamilton with the Crossley reflector had been undertaken by the Columbia University Observatory, New York; and at the request of Professor W. W. Campbell Professor J. K. Rees pushed forward the measurement of eight exposures of the planet and of selected comparison stars, and sent me the measures. Professor F. P. Leavenworth measured seven exposures made at the Observatory of the University of Minnesota, and Professor H. C. Wilson, Director of the Goodsell Observatory, replied that the plate I wanted had been spoiled by passing clouds, and was probably not worth measuring, but promised measures of any other plates that might be useful. To all my sincere thanks are due for the kindness with which they undertook to place at my disposal the material for this second experiment.

4. Meanwhile M. Loewy, President of the *Eros* Commission of the Paris International Conference of 1900, had undertaken to superintend the computation of an eight-figure ephemeris of *Eros* based upon the latest orbit of Professor Millosevich. It was to give, as usual, apparent R.A. and Decl. of the planet, but at my request he promised to print also the separate heliocentric ephemerides of the planet and of the Earth, expressed in rectangular coordinates referred to the mean ecliptic and equinox of 1900·0, which arise in the course of computation of the other ephemeris, and which are of the form most convenient for the method I am using. These ephemerides have recently appeared in the ninth *Paris circular*, and I must not fail to acknowledge my obligations to M. Loewy for the kindness with which he has taken much trouble to forward the trial of a plan very different from that which he himself prefers.

5. We must now reproduce briefly the expressions for the computed place of the planet upon our standard plate, whose centre is given.

Take three rectangular axes, with centre of Sun as origin, directed towards the mean equinox for 1900·0, the point in the ecliptic whose longitude is 90° , and the pole of the ecliptic. At an epoch T , when light left it, the coordinates of the planet on

Q Q 2

this system are U, V, W ; at the epoch $T + \Delta T$ when that light reached the Earth the coordinates of the Earth are f, g, h . These six quantities, $U, V, W; f, g, h$, are found by interpolation in the heliocentric ephemerides referred to above; and $U - f, V - g, W - h$ are the coordinates of the planet at T as seen from the centre of the Earth at $T + \Delta T$.

Suppose now that we take a new system of rectangular axes through the centre of the Earth, parallel to the two axes of rectangular coordinates on the standard plate and to the normal to the plate; and let l_1, m_1, n_1 , &c., be the direction cosines of the new axes with respect to the old. Then the coordinates of the planet with respect to these new axes are

$$L = l_1(U - f) + l_2(V - g) + l_3(W - h)$$

$$M = m_1(U - f) + m_2(V - g) + m_3(W - h)$$

$$N = n_1(U - f) + n_2(V - g) + n_3(W - h)$$

and, omitting for the moment the parallax terms, the coordinates of the planet on the standard plate are $\xi_o = L/N, \eta_o = M/N$.

The procedure adopted in the first paper was therefore as follows: Compute for each exposure $T, T + \Delta T$, and the corresponding interpolation coefficients, and thence the coordinates $U, V, W; f, g, h$.

Compute once for all the nine direction cosines; thence for each exposure the values of L, M, N ; and thence ξ_o, η_o .

Now all this involved a good deal of multiplication by direction cosines. Performed on the arithmometer the work was heavy, but I do not think heavier than in the alternative process of interpolating in the ephemeris for App. R.A. and Decl., and thence transforming to rectangular coordinates on the standard plate. The interpolation in the heliocentric ephemerides is much less troublesome; but without an arithmometer the labour of the whole process would have been prohibitive.

6. It now seems to me that when we have to deal with a large number of exposures with approximately the same centering, or which have been reduced to the same standard plate, an altogether better plan is to start by constructing a "plate ephemeris" for that particular centre.

For the standard plate of 1900 Nov. 9 and 10 the coordinates of whose centre are $\alpha 1^h 57^m 8^s \delta + 54^\circ 22'$ I have made a plate ephemeris as follows:—

f, g, h are the heliocentric ecliptic coordinates of the Earth for successive Berlin midnights taken directly from the ephemeris (*Paris circular*, No. 9, p. 194, where $f = -X, g = -Y, h = -Z$).

U, V, W are the heliocentric ecliptic coordinates of *Eros*, interpolated from the ephemeris of the planet, for successive epochs—Berlin midnight minus the aberration time.

The values of the direction cosines are :—

l_1	-0.48912876	m_1	-0.70890040	n_1	$+0.50814690$
l_2	$+0.80015971$	m_2	-0.13284009	n_2	$+0.58489137$
l_3	-0.34712745	m_3	$+0.69268589$	n_3	$+0.63220945$

Thence we find for each Berlin midnight the values of L , M , N , and thence of ξ_0 , η_0 .

We have now an ephemeris which gives the standard co-ordinates on our plate (except for the parallax displacements which have not yet been put in) of the places of the planet at the times light left it as seen from the Earth at successive Berlin midnights when the light arrived. If we now interpolate in this ephemeris for the mean epochs of our exposures, and calculate the parallax displacements exactly as before (*Monthly Notices*, lxii. p. 30) we have what we want, the computed standard co-ordinates of the planet at the times the light left it, as seen from our position on the Earth at the times the light reached it.

Plate Ephemeris of Eros for Plate Centre, $1^h 57^m 8^s + 54^\circ 22' 0''$ (1900-0).

	ξ_0	η_0
1900.		
Nov. 7	$+0.01472591$	-0.00060344
8	$+0.00947016$	0.00030762
9	$+0.00422788$	0.00027106
10	-0.00098974	0.00049666
11	-0.00617220	0.00098664
12	-0.01130788	-0.00174276

There is here a considerable saving of labour. We save the greater part of the computation of the epoch T and its interpolation coefficients ; of the interpolations for the six quantities $U, V, W ; f, g, h$; of the long multiplications by direction cosines, and of the subsequent divisions by N ; for these operations have to be performed for six epochs only instead of perhaps a hundred. Eventually we have to interpolate for two quantities, standard coordinates instead of App. R.A.'s and Decl. ; and the labour of constructing the plate ephemeris to start with is much less than that of turning a large number of R.A.'s and Decl.s. into standard coordinates, especially when the former can be done on the arithmometer. Finally we are spared the labour of computing and taking out the correction to the planet's place for the aberration of light, which has in the ordinary course been automatically but improperly introduced by the process of reduction (see Mr. Cowell's paper in *The Observatory*, 1900 December, and mine, *Monthly Notices*, lxii. p. 29).

7. The reduction to standard of the Cambridge exposures

has been done in two steps. An exposure near the meridian on Nov. 10 was selected as a "zero exposure," and all the exposures of the series A (guiding on the stars) were first reduced to this by the six constant formulæ (Turner's), with constants derived from the measures of fourteen comparison stars nearly equal to the planet in magnitude. All were then reduced from zero to standard by the constants derived from measures of thirteen meridian standard stars on the zero plate.

In the present reductions I have taken the mean of all the plates of series A reduced to zero as a new zero plate, with the purpose of avoiding systematic errors in the constants of reduction to zero which might arise from accidental errors in the old single zero plate. We will call this the mean zero plate.

The constants of the reduction from mean zero to standard were derived from the seven stars on the plate which have been observed by a great number of observatories as *étoiles de repère* for the *Eros* plates (*Paris circulars* 8 and 9). The number is small for two reasons: first, that the Paris list was selected for the astrographic plates of 2° square, and a number of stars lie just outside the edge of my plates, which cover only $1\frac{1}{2}^\circ$ square; second, that two of the Paris stars are not available, one because it is so bright that adjacent images interfere with one another, the other because it is unfortunately the variable *U Persei*, which was faint at the time that *Eros* passed near it. By bad luck these two stars were especially wanted to give a good determination of the scale value in ξ ; and I am not satisfied that a part of the apparent increase of the tabular error of the ephemeris with the time may not be due to erroneous scale value. This would be fatal to a determination of the parallax by the diurnal method, but is easily seen to have no sensible effect on the present reductions.

8. I have chosen deliberately to retain the six constants in the reduction, that is, to admit a possible difference of scale value in the directions of the two axes of coordinates, rather than to calculate and apply the whole or the unsymmetrical part of the differential refraction, and thus reduce the constants required by theory to four. The argument of those who contend for the latter procedure appears to be that when the unsymmetrical part of the refraction is removed all the remaining necessary corrections ought to be, and therefore are, symmetrical. In reply to this I would urge that in actual fact they frequently are not. Time after time I have found that the asymmetry of the six constants found by the direct reduction, according to Turner, does not correspond at all with the asymmetry produced by the differential refraction. Pending the collection of more material for an examination of this point, one may do either of two things:—use six constants and secure the best fit possible between two plates to be compared, or use four and be satisfied with a fit not the best possible, but the best consistent with the hypothesis of symmetry. I think that there can be no doubt that the former

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of Photographs of Eros.

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is the more prudent course, and am not at present prepared to admit that the use of the six constants involves any loss of weight or of rigour.

9. The following are the details of the photographs used in the present discussion.

Lick. Crossley Reflector. Aperture 3 ft. ; focal l. 17 ft. 6 in.

Plate No.	Epoch of Mid. Exp.	Duration of			No. of Comp. Stars.
		h	m	s	
461 (1)	G.M.T. Nov. 9	15	18	15	6
(5)		15	19	50	6
462 (3)		15	24	5	6
472 (1)	Nov. 10	14	16	15	7
(3)		14	16	57.5	7
480 (1)		14	50	15	7
(3)		14	50	57.5	7

Minneapolis. Refractor. Aperture 10½ in. ; focal l. 11 ft. 3 in.

	Epoch	h	m	s	m	s	
1	Nov. 10	13	52	6	2	0	14
3		14	3	49	3	30	14

Cambridge. Refractor. Aperture 12½ in. ; focal l. 19 ft. 2 in.

	Epoch	h	m	s	m	
246	Nov. 9	15	36	1	3	14
247		15	39	51	2	"
250		16	35	21	3	"
251		16	38	11	2	"
274	Nov. 10	12	39	29	3	"
275		12	42	24	2	"
278		13	37	24	3	"
279		13	40	19	2	"
282		14	40	54	3	"
283		14	44	24	2	"

After a preliminary reduction of the Lick and Minneapolis measures to the scale of the Cambridge plates the equations of condition were formed to reduce each exposure to the Cambridge mean zero plate, the six constants were found for each exposure, and the measures of the planet reduced to Cambridge mean zero, and thence to standard by the constants derived from the reduction of the mean zero plate to standard with the new places of the *étoiles de repère* from the Paris circulars. These standard places of the planet are next expressed in parts of the radius of projection, and thenceforward the unit to be employed

is 1×10^{-7} of the radius. It has been assumed for the present that the weight of each plate is the same.

Since we are concerned more with methods than with numerical results, it is unnecessary to print the details of the measures and reductions.

10. Neither is it necessary to follow in detail the steps required to form the equations of condition. The tabular places of the planet were interpolated from the "plate ephemeris"; the parallax factors were found as in the previous paper (*Monthly Notices*, vol. lxii. p. 30); the tabular places were corrected for parallax with an assumed value of the solar parallax, and compared with the observed places. Finally means were taken of pairs of observations close together, which are then treated as one observation.

If $\Delta\pi$ is the correction required to the assumed value of the solar parallax, $\Delta\xi_0$ and $\Delta\eta_0$ the corrections to the plate ephemeris supposed constant for the two short periods under discussion, we have the following equations of condition:—

Nov. 9.					Residuals.
Camb. 246,7	$+1.56 \Delta\pi$	$-1 \Delta\xi_0$	$0 \Delta\eta_0$	$-90 = 0$	-3
	$+0.86$	0	-1	$+5$	$+7$
250,1	$+1.62$	-1	0	-80	$+7$
	$+1.19$	0	-1	-16	-14
Lick 461 (1, 5)	-1.68	-1	0	-85	-1
	-0.11	0	-1	$+5$	$+6$
462 (3)	-1.65	-1	0	-87	-2
	-0.15	0	-1	$+2$	$+2$

The last two equations, being dependent upon one exposure only, have been given half weight.

Nov. 10.					Residuals.
Camb. 274,5	$+0.84 \Delta\pi$	$-1 \Delta\xi_0$	$0 \Delta\eta_0$	$-72 = 0$	$+1$
	$+0.08$	0	-1	-27	$+3$
278,9	$+1.17$	-1	0	-69	$+3$
	$+0.29$	0	-1	-28	$+2$
282,3	$+1.44$	-1	0	-73	-2
	$+0.59$	0	-1	-27	$+2$
Lick 472 (1, 3)	-1.96	-1	0	-77	$+5$
	$+0.27$	0	-1	-31	-1
480 (1, 3)	-1.81	-1	0	-83	-1
	$+0.04$	0	-1	-31	0
Minn. 1, 3	-1.30	-1	0	-86	-6
	-0.01	0	-1	-36	-5

The normal equations derived from these are :—

$$\begin{array}{rcll}
 \text{Nov. 9.} & +11.43 \Delta\pi & -0.67 \Delta\xi_0 & -1.86 \Delta\eta_0 & -70 = 0 \\
 & -0.67 \Delta\pi & +3.50 \Delta\xi_0 & 0.00 \Delta\eta_0 & +299 = 0 \\
 & -1.86 \Delta\pi & 0.00 \Delta\xi_0 & +3.50 \Delta\eta_0 & +5 = 0 \\
 \text{Nov. 10.} & +13.47 \Delta\pi & +1.62 \Delta\xi_0 & -1.26 \Delta\eta_0 & +131 = 0 \\
 & +1.62 \Delta\pi & +6.00 \Delta\xi_0 & 0.00 \Delta\eta_0 & +460 = 0 \\
 & -1.26 \Delta\pi & 0.00 \Delta\xi_0 & +6.00 \Delta\eta_0 & +180 = 0
 \end{array}$$

Eliminating successively $\Delta\xi_0$ and $\Delta\eta_0$ from each of these sets we obtain two equations for $\Delta\pi$.

$$\begin{array}{rcl}
 \text{From Nov. 9.} & +10.31 \Delta\pi & = +10 \\
 \text{Nov. 10.} & +12.77 \Delta\pi & = -45.
 \end{array}$$

Assuming that the p.e. of an equation of weight unity is the same on each night we add these together, and find

$$\Delta\pi = -1.5 = -0''.031.$$

If we carry out the solutions separately for each day we obtain

$$\begin{array}{rcll}
 \text{For November 9} & \Delta\xi_0 = -85 & \Delta\eta_0 = -1 & \Delta\pi = +0.97 \\
 10 & \Delta\xi_0 = -76 & \Delta\eta_0 = -31 & \Delta\pi = -3.52
 \end{array}$$

and substituting back in the equations of condition we obtain the residuals given in the last column.

From these we find

P.E. of one equation of weight unity

$$\begin{array}{rcl}
 \text{November 9} & \pm 5.6 & = \pm 0''.11 \\
 10 & \pm 2.4 & = \pm 0''.05
 \end{array}$$

The great difference between the apparent weights for the two nights is caused almost entirely by the discrepant Cambridge observation in η on November 9 (the fourth equation of condition). On the other hand, perhaps the observations of November 10 are unusually accordant. We may take for the present the p.e. of one equation of weight unity, derived from the mean of a pair of exposures, as about $\pm 0''.07$; and it is scarcely worth while to examine minutely the probable errors of the quantities derived from so few equations.

The differences between the corrections to the tabular places on November 9 and 10 come out rather large, and it is not possible with this small amount of material to say how far they are real. But they emphasise the necessity for dividing the equations, as has been done here, into groups derived from observations nearly simultaneous.

I have refrained from giving the assumed value of the parallax to which these observations give the above correction, because no good purpose can be served by publishing preliminary results which have but little weight.

II. In summing up the conclusions which may be drawn from these two experimental reductions of *Eros* photographs I would submit.

(1) That the adopted form of ephemeris in rectangular co-ordinates has been found convenient, and may be used in future definitive reductions.

(2) That the direct comparison of simultaneous photographs by linear reductions is the most convenient method of dealing with the *Eros* plates. If the same comparison stars are used, it is not required that their meridian places should all be known; if the meridian places of the comparatively few standard stars are not quite accurate, the resulting parallax is scarcely affected; if the ephemeris needs correction by terms varying with the time, practically no harm is done. In fact, by working in this way we can find the parallax by methods which are differential, and do not involve absolute places, except in a secondary degree. The gain is obvious.

(3) The smallness of the probable error of an equation of condition is satisfactory evidence of the value of photographic methods for finding the solar parallax. If the results of a larger series come out as well as those that have been discussed here, that is to say, if the p.e. of an equation of condition derived from the mean of two exposures is about $\pm 0''.07$, we shall not want to discuss the whole of the existing material to obtain a value of the solar parallax with a p.e. $\pm 0''.005$, which is the value Gill finds for the p.e. of his determination from the heliometer observations of *Victoria*, *Iris*, and *Sappho*. It will be a matter of great interest to see whether the photographic observations of *Eros* confirm this result, which has just been adopted by the Nautical Almanacs, or bring out a smaller value, more in accordance with that required by the planetary theory, and by recent determinations of the constant of aberration combined with the velocity of light.

The sooner this can be done the better. I think, therefore, that our next step should be to extend the limits of these reductions so as to include enough material to give us a value of the solar parallax with a p.e. less than $\pm 0''.005$. We propose to measure next all our Cambridge plates between 1900 November 7 and 15 inclusive, and to combine them with as many simultaneous exposures made at other observatories as may be available. To this end I should like to propose to seven or eight observatories spread over as long an arc of longitude as possible that we should agree upon a common list of comparison stars, and measure up all our plates in that period of nine days. If we can concentrate our attention upon this block of the work the measurement is not so formidable a task as to consume a great

many months. Then when the measures are made we might publish or interchange this raw material and proceed to its discussion.

It should be possible thus to settle in two or three years the question whether or not *Eros* is going to give us the same value of the solar parallax as did the three other less happily situated minor planets. That, I take it, is the main point. If it does not, and we have to proceed eventually to the discussion of the whole mass of material, we shall have gained experience and lost no time by concentrating our energies at first upon a few days' plates.

I must add that my sincere thanks are due to the directors of the Lick Observatory, of the Columbia University Observatory, and of the Oxford University Observatory, who have already cordially agreed to enter into this plan.

Cambridge Observatory :
1902 June 9.

Reductions of photographs of Swift's Comet (a 1899), taken at the Cambridge Observatory, with a portrait lens. By L. N. G. Filon, M.A., B.Sc.

1. In the summer of 1899 I was working at the Cambridge Observatory, and I undertook to help Mr. A. R. Hinks in taking some photographic observations of Swift's Comet *a* 1899, which was visible during May and the beginning of June as a telescopic object.

The plates taken then had perforce to remain unmeasured for a long time, as the present measuring machine of Cambridge Observatory was still in course of construction, and it was nearly eighteen months before it was ready for use. Stress of other work, however, and the fact that the machine was urgently needed at the Observatory for the measurement of the *Eros* parallax plates prevented my attending at the time to the reduction of these observations.

In February last the machine was available, and the plates capable of measurement were then measured and reduced. The results of the reduction are presented in this paper.

The photographs of the comet were taken with a portrait lens, of about 30 inches focal length and 5 inches aperture, stopped down to work at $f/8$. The portrait lens and its camera, for the loan of which I understand the Observatory was indebted to Professor Turner, were fixed to the tube of the Northumberland Equatorial, which was used as a guiding telescope. The camera was placed at the object-glass end of the telescope. An exposing shutter could be opened and shut from the eye end by pulling a string attached for the purpose.